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LOW REGULARITY SOLUTIONS TO HIGHER-ORDER HARTREE-FOCK EQUATIONS WITH UNIFORM BOUNDS

Changhun Yang

Abstract. In this paper, we consider the higher-order Hartree-Fock equations. The higher-order linear Schrödinger equation was introduced in [5] as the formal finite Taylor expansion of the pseudo-relativistic linear Schrödinger equation. In [13], the authors established global-in-time Strichartz estimates for the linear higher-order equations which hold uniformly in the speed of light $c \geq 1$ and as their applications they proved the convergence of higher-order Hartree-Fock equations to the corresponding pseudo-relativistic equation on arbitrary time interval as c goes to infinity when the Taylor expansion order is odd. To achieve this, they not only showed the existence of solutions in L^2 space but also proved that the solutions stay bounded uniformly in c.

We address the remaining question on the convergence of higher-order Hartree-Fock equations when the Taylor expansion order is even. The distinguished feature from the odd case is that the group velocity of phase function would be vanishing when the size of frequency is comparable to c. Owing to this property, the kinetic energy of solutions is not coercive and only weaker Strichartz estimates compared to the odd case were obtained in [13]. Thus, we only manage to establish the existence of local solutions in H^s space for $s > \frac{1}{3}$ on a finite time interval $[-T, T]$, however, the time interval does not depend on c and the solutions are bounded uniformly in c. In addition, we provide the convergence result of higher-order Hartree-Fock equations to the pseudo-relativistic equation with the same convergence rate as the odd case, which holds on $[-T, T]$.

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1. Introduction

We consider the higher-order Hartree-Fock equations of $N(\in\mathbb{N})$ particles

$$\text{(hHF)}\qquad \begin{cases} i\hbar\partial_t\phi_k^{(c)} = \mathcal{H}_J^{(c)}\phi_k^{(c)} + H(\phi_k^{(c)}) - F_k(\phi_k^{(c)}), \quad k=1,2,...,N,\\ \phi_k^{(c)}(0,x) = \phi_{k,0} \end{cases}$$

where $\phi_k^{(c)} = \phi_k^{(c)}(t,x) : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{C}$, $m>0$ represents the particle mass, $\hbar$ is the reduced Plank constant, and $c \geq 1$ denotes the speed of light. The linear differential operator is given by

$$\mathcal{H}_J^{(c)} = -\sum_{j=1}^{J} \frac{\alpha(j)\hbar^{2j}}{m^{2j-1}c^{2j-2}}\Delta^j, \qquad \alpha(j) = \frac{(2j-2)!}{j!(j-1)!2^{2j-1}} \quad (j\geq 1),$$

and the Hartree-Fock nonlinear terms are given by

$$H(\phi_k^{(c)}) = \sum_{\ell=1}^{N}\left(\frac{\kappa}{|x|} * |\phi_\ell^{(c)}|^2\right)\phi_k^{(c)},$$
$$F_k(\phi_k^{(c)}) = \sum_{\ell=1,\ell\neq k}^{N}\left(\frac{\kappa}{|x|} * (\overline{\phi_\ell^{(c)}}\phi_k^{(c)})\right)\phi_\ell^{(c)}.$$

A real constant κ determines the strength of self-interaction among quantum particles; it is repulsive if $\kappa>0$, and attractive if $\kappa<0$. The mass and energy of solutions are conserved as time evolves which are defined as

$$\text{(1.1)}\qquad \begin{aligned} \mathcal{M}(t) &= \sum_{k=1}^{N}\|\phi_k^{(c)}\|_{L^2(\mathbb{R}^3)}^2,\\ \mathcal{E}(t) &= \sum_{k=1}^{N}\Big\{\frac{1}{2}\big\langle\mathcal{H}_J^{(c)}\phi_k^{(c)},\phi_k^{(c)}\big\rangle + \frac{1}{4}\big\langle H\big(\phi_k^{(c)}\big) - F_k\big(\phi_k^{(c)}\big),\phi_k^{(c)}\big\rangle\Big\}, \end{aligned}$$

where $\langle\, ,\, \rangle$ denotes the complex inner product in $L^2(\mathbb{R}^3)$.

The higher-order linear Schrödinger equation was introduced in [5]:

$$\text{(hLS)}\qquad i\hbar\partial_t\phi^{(c)} = \mathcal{H}_J^{(c)}\phi^{(c)}$$

as the formal approximation of the *pseudo-relativistic* (or *semi-relativistic*) linear Schrödinger equations

$$\text{(pLS)}\qquad i\hbar\partial_t\psi^{(c)} = \mathcal{H}^{(c)}\psi^{(c)},$$

where $\psi^{(c)} = \psi^{(c)}(t,x) : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{C}$ is the wave function and the nonlocal operator

$$\mathcal{H}^{(c)} = \sqrt{m^2c^4 - c^2\hbar^2\Delta} - mc^2$$

is the Fourier multiplier of symbol $\sqrt{m^2c^4 + c^2\hbar^2|\xi|^2} - mc^2$. Indeed, in the non-relativistic regime $|\xi| \ll \frac{mc}{\hbar}$, the Taylor series expansion yields

$$\begin{aligned}\sqrt{\hbar^2c^2|\xi|^2 + m^2c^4} - mc^2 &= mc^2\left(\sqrt{1+\tfrac{\hbar^2|\xi|^2}{m^2c^2}} - 1\right)\\ &= \tfrac{\hbar^2|\xi|^2}{2m} - \tfrac{\hbar^4|\xi|^4}{8m^3c^2} + \cdots \approx \tfrac{\hbar^2|\xi|^2}{2m}.\end{aligned} \tag{1.2}$$

Note that higher-order models include the non-relativistic Schrödinger equation $i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\Delta\psi$ and the fourth-order equation $i\hbar\partial_t\psi = (-\frac{\hbar^2}{2m}\Delta - \frac{\hbar^4}{8m^3c^2}\Delta^2\big)\psi$. In [4], the authors showed that the higher-order linear flow provides a more accurate approximation as $c\to\infty$ as long as the regularity of data is sufficiently given [4, Theorem A.1], precisely,

$$\|e^{it\mathcal{H}^{(c)}}\psi_0 - e^{it\mathcal{H}_J^{(c)}}\psi_0\|_{L^2(\mathbb{R}^d)} \le \frac{2T}{\hbar}\frac{\alpha(J+1)}{m^{2J+1}c^{2J}}\|\psi_0\|_{H^{2J+2}(\mathbb{R}^d)}. \tag{1.3}$$

The question on approximation via higher-order equation as $c\to\infty$ can be extended to nonlinear problem. We consider the pseudo-relativistic Hartree-Fock equation

$$i\hbar\partial_t\psi_k^{(c)} = \mathcal{H}^{(c)}\psi_k^{(c)} + H(\psi_k^{(c)}) - F_k(\psi_k^{(c)}),\quad k=1,2,...,N, \tag{pHF}$$

where $\psi_k^{(c)} = \psi_k^{(c)}(t,x) : \mathbb{R}\times\mathbb{R}^3\to\mathbb{C}$, which describes the mean-field dynamics of relativistic fermion particles. For the rigorous derivation of the pseudo-relativistic models we refer to [16, 17, 3] and for the dynamics of the system we refer to [11, 7, 12, 14]. The higher-order Hartree-Fock equations (hHF) was introduced in [4] as the formal approximation of (pHF). Proving the approximation of pseudo-relativistic models by the non-relativistic models not only justifies consistency of the relativistic modification but also verifies that non-relativistic models are good approximations to the pseudo-relativistic models, which are computationally extremely expensive due to the presence of the non-local operator $\mathcal{H}^{(c)}$.

When J is odd, the authors in [13] established that the nonlinear solutions to (hHF) indeed converge to solutions to (pHF) on any time interval as $c\to\infty$ with a more accurate convergence rate as J grows.

Theorem 1.1 (Theorem 1.7 in [13]). *Let $J \in 2\mathbb{N}-1$ and $c \geq 1$. Suppose that $\Psi_0 = \{\psi_{k,0}\}_{k=1}^N \in H^{\frac{1}{2}}(\mathbb{R}^3;\mathbb{C}^N)$. If $\kappa < 0$, we further assume that $\|\Psi_0\|_{H^{\frac{1}{2}}(\mathbb{R}^3;\mathbb{C}^N)}$ is sufficiently small. Let $\Psi^{(c)}(t) = \{\psi_k^{(c)}(t)\}_{k=1}^N \in C(\mathbb{R};H^{\frac{1}{2}}(\mathbb{R}^3;\mathbb{C}^N))$ be the global solution to* (pHF) *with initial data Ψ_0, and let $\Phi^{(c)}(t) = \{\phi_k^{(c)}(t)\}_{k=1}^N \in C(\mathbb{R};H^{\frac{1}{2}}(\mathbb{R}^3;\mathbb{C}^N))$ be the global solution to* (hHF) *with the same initial data. Then, there exist $A, B > 0$, depending on $\|\Psi_0\|_{H^{\frac{1}{2}}(\mathbb{R}^3;\mathbb{C}^N)}$ but independent of $c \geq 1$ and J, such that*

$$\|\Phi^{(c)}(t) - \Psi^{(c)}(t)\|_{L^2(\mathbb{R}^3;\mathbb{C}^N)} \leq Ac^{-\frac{J}{2(J+1)}}e^{Bt}. \tag{1.4}$$

The methodology to prove the approximation as $c \to \infty$ is quite standard. As the first step, it is necessary to find the solutions which have uniform bounds with respect to $c \geq 1$. And then, we write the solutions in Duhamel formulae and measure the difference in $L^2(\mathbb{R}^3)$. The convergence rate comes from the Taylor expansion of symbol and the regularity gap from $H^{\frac{1}{2}}(\mathbb{R}^3)$ where the initial data is given. In the proof, it is crucial to employ the inequalities with bounds independent of $c \geq 1$. In [13], to prove the Theorem 1.1, the authors indeed established the global-in-time Strichartz estimates which hold uniformly in $c \geq 1$. As their applications, they proved the local existence of solutions to (hHF) in $L^2(\mathbb{R}^3)$ which have uniform bounds in $c \geq 1$ and extended the local solutions to global ones with the help of the conservation of mass and energy (1.1).

In this paper, we consider (hHF) when the Taylor expansion order J is even. The distinguished feature from the odd order case is that the dispersion effect is unsatisfactory. Contrary to the odd case when the phase function is essentially comparable to the Laplacian in $|\xi| \leq \frac{2m}{\hbar}c$, the group velocity of phase function for even J vanishes when the frequency is close to c. More precisely, the phase function is radial and given by

$$\widehat{\mathcal{H}_J^{(c)}\phi}(\xi) = \omega_J^{(c)}(|\xi|)\widehat{\phi}(\xi),$$

where

$$\omega_J^{(c)}(r) = \sum_{j=1}^{J} \frac{(-1)^{j+1}(2j-2)!\hbar^{2j-1}}{(j-1)!j!(2m)^{2j-1}c^{2j-2}}r^{2j}, \text{ for } r \geq 0,$$

where one readily verifies that $\frac{d}{dr}\omega_J^{(c)}(r_*) = 0$ for some $\frac{m}{\hbar}c < r_* < \frac{2m}{\hbar}c$. Due the this property, only the uniform-in-c *weaker* Strichartz estimates were established in [13] (see Lemma 2.1 below). Additionally, it should be emphasized that the coerciveness of kinetic energy breaks down.

We establish the local well-posendess of (hHF) in $H^s(\mathbb{R}^3)$ for $s > \frac{1}{3}$ by employing the uniform-in-c *weaker* Strichartz estimates. The time interval where the solutions exist does not depend on $c > 1$ and the local solutions are bounded on the interval uniformly in $c > 1$. Due to the lack of coerciveness of kinetic energy, unfortunately, the local solutions cannot be extended globally. Let us recall the definition of Sobolev spaces: $W^{s,p} = (1-\Delta)^{-\frac{s}{2}}L^p$ for $p \geq 1$ and $s \geq 0$.

THEOREM 1.2 (Local H^s solutions with uniform-in-c bounds). *Let $c \geq 1$ and $J \in 2\mathbb{N}$. Suppose that $\Phi_0 = \{\phi_{k,0}\}_{k=1}^N \in H^s(\mathbb{R}^3;\mathbb{C}^N)$ for $s > \frac{1}{3}$. Then, there exists a time $T > 0$, independent of $c \geq 1$, such that* (hHF) *has a unique solution $\Phi^{(c)}(t) = \{\phi_k^{(c)}(t)\}_{k=1}^N$ in the class*

$$\Phi^{(c)} \in C([-T,T];H^s(\mathbb{R}^3;\mathbb{C}^N)) \cap L_t^q([-T,T];W^{2\epsilon,r}(\mathbb{R}^3;\mathbb{C}^N)),$$

where $0 < \epsilon < \min(\frac{3s-1}{4}, \frac{1}{2})$ and $(q,r) = (\frac{12}{1-2\epsilon}, \frac{3}{1+\epsilon})$ is an even-admissible pair. Moreover, the solution satisfies

$$\|\Phi^{(c)}\|_{C([-T,T];H^s(\mathbb{R}^3;\mathbb{C}^N))} + \|\Phi^{(c)}\|_{L_t^q([-T,T];W^{2\epsilon,r}(\mathbb{R}^3;\mathbb{C}^N))} \lesssim \|\Phi_0\|_{H^s(\mathbb{R}^3;\mathbb{C}^N)} \tag{1.5}$$

where the implicit constant is independent of $c \geq 1$.

REMARK 1.3. In Theorem 1.2, for example, when $s = \frac{1}{2}$, any $0 < \epsilon < \frac{1}{8}$ will be working.

REMARK 1.4. We expect that the regularity assumption $s > \frac{1}{3}$ might be lowered once we can improve the *weaker* Strichartz estimates in (2.2).

REMARK 1.5. Employing the local-in-time Strichartz estimates [4, Lemma 4.3], the authors proved the existence of global solutions to higher-order Hartree-Fock equations (hHF) in $L^2(\mathbb{R}^3)$ for all $J \in \mathbb{N}$ [4, Theorem 4.9]. However, the employed Strichartz estimates are not uniform in the speed of light $c \geq 1$, thus the boundedness of solutions as c goes to infinity is not guaranteed even in an arbitrarily small time interval.

Next, we establish the convergence of (hHF) to (pHF) when J is even which corresponds to Theorem 1.1 for odd J.

Theorem 1.6. *Let $J \in 2\mathbb{N}$ and $c \geq 1$. Suppose that $\Psi_0 = \{\psi_{k,0}\}_{k=1}^N \in H^s(\mathbb{R}^3;\mathbb{C}^N)$ for $s \geq \frac{1}{2}$. Then there exists a time $T > 0$ depending on $\|\Psi_0\|_{H^s(\mathbb{R}^3;\mathbb{C}^N)}$, but independent of $c > 1$, such that $\Psi^{(c)}(t) = \{\psi_k^{(c)}(t)\}_{k=1}^N \in C(\mathbb{R}; H^s([-T,T];\mathbb{C}^N))$ is the local solution to* (pHF) *with initial data Ψ_0, and let $\Phi^{(c)}(t) = \{\phi_k^{(c)}(t)\}_{k=1}^N \in C([-T,T]; H^s(\mathbb{R}^3;\mathbb{C}^N))$ be the local solution to* (hHF) *with the same initial data. Then, there exist $A > 0$, depending on $\|\Psi_0\|_{H^s(\mathbb{R}^3;\mathbb{C}^N)}$ but independent of $c \geq 1$, such that*

$$\sup_{t\in[-T,T]} \|\Phi^{(c)}(t) - \Psi^{(c)}(t)\|_{L^2(\mathbb{R}^3;\mathbb{C}^N)} \leq Ac^{-\frac{Js}{J+1}}. \tag{1.6}$$

Nevertheless we obtain uniformly bounded solutions to (hHF) in $H^s(\mathbb{R}^3)$ for $s > \frac{1}{3}$, the regularity assumption $s \geq \frac{1}{2}$ in Theorem 1.6 is required to obtain uniformly bounded solutions to (pHF). This assumption is same as given in Theorem 1.1. See Proposition 3.1 and the comment in Remark 3.2. We also obtain the same converge rate as in Theorem 1.1 when $s = \frac{1}{2}$, but the convergence is valid only for the finite time interval $[-T,T]$.

Remark 1.7. The argument in our paper can be directly applied to the following Hartree equation to provide the similar approximation results

$$\begin{cases} i\hbar\partial_t\phi_k^{(c)} = \mathcal{H}_J^{(c)}\phi_k^{(c)} + H(\phi_k^{(c)}), \\ i\hbar\partial_t\psi_k^{(c)} = \mathcal{H}^{(c)}\psi_k^{(c)} + H(\psi_k^{(c)}), \end{cases} \quad k = 1,2,...,N. \tag{1.7}$$

The limit behavior of stationary states to the higher-order Hartree equations (1.7) was studied in [9]. We also refer to [15, 10, 8] for non-relativistic limit (corresponding $J = 1$) of pseudo-relativistic Hartree equations. Also, similar non-relativistic convergence can be formulated for other types of relativistic models, and there have been numerous results on this direction. We refer to [18, 19] for the non-relativistic limit from the Klein-Gordon and the Dirac equation to the nonlinear Schrödinger equation. Also, we refer to [1, 2] for the convergence from the Dirac-Maxwell (resp., Klein-Gordon-Maxwell) system to the Vlasov-Poisson (resp., Schrödinger-Poisson) system. Finally, we refer to [20, 21, 22] for the non-relativistic limits of the Dirac-Maxwell, Klein-Gordon-Maxwell and Klein-Gordon-Zakharov systems.

2. Preliminaries

Let us recall from [13, Theorem 1.4] the uniform-in-c Strichartz estimates for higher-oder linear equations (hLS) when J is even.

THEOREM 2.1 (Strichartz estimates for (hLS) : even case). *Let $J \in 2\mathbb{N}$. Then, there exists $A > 0$, independent of $c \geq 1$, such that for an even-admissible pair (q, r),*

$$2 \leq q \leq \infty, \quad 2 \leq r < \infty, \quad \frac{2}{q} + \frac{1}{r} = \frac{1}{2} \tag{2.1}$$

we have

$$\begin{aligned} \left\| e^{-it\mathcal{H}_J^{(c)}} \psi_0 \right\|_{L_t^q(\mathbb{R};L_x^r(\mathbb{R}^3))} &\leq A \left(\frac{m}{\hbar}\right)^{\frac{1}{q}} \|\psi_0\|_{\dot{H}^{\frac{4}{q}}(\mathbb{R}^3)}, \\ \left\| \int_0^t e^{-i(t-s)\mathcal{H}_J^{(c)}} F(s) ds \right\|_{L_t^q(\mathbb{R};L_x^r(\mathbb{R}^d))} &\leq A \left(\frac{m}{\hbar}\right)^{\frac{1}{q}} \|F\|_{L_t^1(\mathbb{R};H^{\frac{4}{q}}(\mathbb{R}^3))}. \end{aligned} \tag{2.2}$$

Next, we introduce key inequalities to handle the Hartree-Fock nonlinear term.

LEMMA 2.2. *Let $s \geq \frac{1}{2}$ and $0 < \epsilon \ll 1$. Then, we have*

$$\left\| |x|^{-1} * (f_1 f_2) \right\|_{L^\infty(\mathbb{R}^3)} \lesssim \|f_1\|_{H^s(\mathbb{R}^3)} \|f_2\|_{H^s(\mathbb{R}^3)}, \tag{2.3}$$

and

$$\begin{aligned} &\left\| |x|^{-1} * (f_1 f_2) \right\|_{L^\infty(\mathbb{R}^3)} \\ &\quad \lesssim \|f_1\|_{L^{\frac{6}{2-\epsilon}}(\mathbb{R}^3)}^{\frac{1}{2}} \|f_1\|_{L^{\frac{6}{2+\epsilon}}(\mathbb{R}^3)}^{\frac{1}{2}} \|f_2\|_{L^{\frac{6}{2-\epsilon}}(\mathbb{R}^3)}^{\frac{1}{2}} \|f_2\|_{L^{\frac{6}{2+\epsilon}}(\mathbb{R}^3)}^{\frac{1}{2}}. \end{aligned} \tag{2.4}$$

For the proof, we refer to [10, Lemma 3.2] and [6, Lemma 2.3], respectively. From (2.3) and the fractional Leibniz rule, one immediately has that

$$\begin{aligned} &\left\| \left(|x|^{-1} * (f_1 f_2)\right) f_3 \right\|_{H^s(\mathbb{R}^3)} \\ &\quad \lesssim \|f_1\|_{H^s(\mathbb{R}^3)} \|f_2\|_{H^s(\mathbb{R}^3)} \|f_3\|_{H^s(\mathbb{R}^3)}, \quad \text{for } s \geq \frac{1}{2}. \end{aligned} \tag{2.5}$$

Using the Hardy-Littlewood-Sobolev inequality, one verifies that

$$\left\| \left(|x|^{-1} * (f_1 f_2)\right) f_3 \right\|_{L^2(\mathbb{R}^3)} \lesssim \|f_1\|_{L^3(\mathbb{R}^3)} \|f_2\|_{L^2(\mathbb{R}^3)} \|f_3\|_{L^3(\mathbb{R}^3)}. \tag{2.6}$$

3. Proof of Main theorem

3.1. Proof of Theorem 1.2: Local H^s solutions with uniform-in-c bounds

By time reverse property, let $I = [0,T]$ be a sufficiently small interval to be chosen later. Duhamel's principle ensures that the solution to (hHF) is equivalent to the following integral formula:

$$\Gamma_k(\Phi^{(c)}) = e^{-it\mathcal{H}_J^{(c)}}\phi_{k,0} - i\int_0^t e^{-i(t-s)\mathcal{H}_J^{(c)}}\left(H(\phi_k^{(c)}) - F_k(\phi_k^{(c)})\right)(s)\, ds.$$

Then, the standard argument shows the map Γ is contractive in the ball

$$X_{I,A} = \Big\{\Phi^{(c)} \in C_t(I; H^s(\mathbb{R}^3;\mathbb{C}^N)) \cap L_t^{\frac{12}{1-2\epsilon}}(I; W^{2\epsilon,\frac{3}{1+\epsilon}}(\mathbb{R}^3;\mathbb{C}^N)) : \|\Phi^{(c)}\|_{X_I} \le 2A\|\Phi_0\|_{H^s(\mathbb{R}^3;\mathbb{C}^N)}\Big\},$$

$$\|\Phi^{(c)}\|_{X_I} := \|\Phi^{(c)}\|_{C_t(I;H^s(\mathbb{R}^3\ \mathbb{C}^N))} + \|\Phi^{(c)}\|_{L_t^{\frac{12}{1-2\epsilon}}(I;W^{2\epsilon,\frac{3}{1+\epsilon}}(\mathbb{R}^3;\mathbb{C}^N))}.$$

Here, $A > 0$ denotes the uniform-in-c constant in Theorem 2.1. Indeed, by unitarity and Strichartz estimates (2.2), since $\epsilon < \frac{3s-1}{4}$, we have

$$\begin{aligned} &\|\Gamma(\Phi^{(c)})\|_{C_t(I;H^s(\mathbb{R}^3;\mathbb{C}^N))} + \|\Gamma(\Phi^{(c)})\|_{L_t^{\frac{12}{1-2\epsilon}}(I;W^{2\epsilon,\frac{3}{1+\epsilon}}(\mathbb{R}^3;\mathbb{C}^N))} \\ &\le A\|\Phi_0\|_{H^s(\mathbb{R}^3;\mathbb{C}^N)} \\ &\quad + 2A|\lambda|\sum_{k,\ell=1}^N \Big\{\Big\|\Big(|x|^{-1} * |\phi_\ell^{(c)}|^2\Big)\phi_k^{(c)}\Big\|_{L_t^1(I;H^s(\mathbb{R}^3))} \\ &\qquad\qquad + \Big\|\Big(|x|^{-1} * (\overline{\phi_\ell^{(c)}}\phi_k^{(c)})\Big)\phi_\ell^{(c)}\Big\|_{L_t^1(I;H^s)(\mathbb{R}^3)}\Big\}. \end{aligned}$$

We only estimate the latter term in the last line, then the same argument applies to the former term. Applying the fractional Leibniz rule and (2.4), we estimate

$$\Big\|\Big(|x|^{-1} * (\overline{\phi_\ell^{(c)}}\phi_k^{(c)})\Big)\phi_\ell^{(c)}\Big\|_{H^s(\mathbb{R}^3)} \lesssim$$

$$\Big\||x|^{-1} * \big(\overline{\phi_\ell^{(c)}}\phi_k^{(c)}\big)\Big\|_{L^\infty(\mathbb{R}^3)}\|\phi_\ell^{(c)}\|_{H^s(\mathbb{R}^3)} \tag{3.1}$$

$$+ \Big\||x|^{-1} * \big(\overline{\phi_\ell^{(c)}}\phi_k^{(c)}\big)\Big\|_{W^{s,\frac{6}{1+2\epsilon}}(\mathbb{R}^3)}\|\phi_\ell^{(c)}\|_{L^{\frac{3}{1-\epsilon}}(\mathbb{R}^3)}. \tag{3.2}$$

Using (2.4) and the Sobolev embedding, we bound (3.1) by

$$\|\phi_\ell^{(c)}\|^{\frac{1}{2}}_{L^{\frac{3}{1-\epsilon}}(\mathbb{R}^3)}\|\phi_\ell^{(c)}\|^{\frac{1}{2}}_{L^{\frac{3}{1+\epsilon}}(\mathbb{R}^3)}\|\phi_k^{(c)}\|^{\frac{1}{2}}_{L^{\frac{3}{1-\epsilon}}(\mathbb{R}^3)}\|\phi_k^{(c)}\|^{\frac{1}{2}}_{L^{\frac{3}{1+\epsilon}}(\mathbb{R}^3)}\|\phi_\ell^{(c)}\|_{H^s(\mathbb{R}^3)}$$
$$\lesssim \|\phi_\ell^{(c)}\|_{W^{2\epsilon,\frac{3}{1+\epsilon}}(\mathbb{R}^3)}\|\phi_k^{(c)}\|_{W^{2\epsilon,\frac{3}{1+\epsilon}}(\mathbb{R}^3)}\|\phi_\ell^{(c)}\|_{H^s(\mathbb{R}^3)}.$$

Applying the Hardy-Littlewood-Sobolev inequality and Sobolev embedding, we bound (3.2) by

$$\Big(\|\phi_\ell^{(c)}\|_{L^{\frac{3}{1+\epsilon}}(\mathbb{R}^3)}\|\phi_k^{(c)}\|_{H^s(\mathbb{R}^3)}+\|\phi_k^{(c)}\|_{L^{\frac{3}{1+\epsilon}}(\mathbb{R}^3)}\|\phi_\ell^{(c)}\|_{H^s(\mathbb{R}^3)}\Big)\|\phi_\ell^{(c)}\|_{L^{\frac{3}{1-\epsilon}}(\mathbb{R}^3)}$$
$$\lesssim \|\phi_\ell^{(c)}\|^2_{W^{2\epsilon,\frac{3}{1+\epsilon}}(\mathbb{R}^3)}\|\phi_k^{(c)}\|_{H^s(\mathbb{R}^3)}$$
$$+\|\phi_\ell^{(c)}\|_{W^{2\epsilon,\frac{3}{1+\epsilon}}(\mathbb{R}^3)}\|\phi_k^{(c)}\|_{W^{2\epsilon,\frac{3}{1+\epsilon}}(\mathbb{R}^3)}\|\phi_\ell^{(c)}\|_{H^s(\mathbb{R}^3)},$$

where we used that

$$\|f\|_{L^{\frac{3}{1+\epsilon}}(\mathbb{R}^3)}\lesssim\|f\|_{W^{\sigma,\frac{3}{1+\epsilon}}(\mathbb{R}^3)},\qquad \text{for } \sigma\ge 0.$$

Then, by using the Hölder inequality in time, we obtain

$$\Big\|\Big(|x|^{-1}*(\overline{\phi_\ell^{(c)}}\phi_k^{(c)})\Big)\phi_\ell^{(c)}\Big\|_{L^1_t(I;H^s(\mathbb{R}^3))}$$
$$\lesssim T^{\frac{5+2\epsilon}{6}}\|\phi_\ell^{(c)}\|_{L_t^{\frac{12}{1-2\epsilon}}W^{2\epsilon,\frac{3}{1+\epsilon}}}\Big(\|\phi_\ell^{(c)}\|_{L_t^{\frac{12}{1-2\epsilon}}W^{2\epsilon,\frac{3}{1+\epsilon}}}+\|\phi_k^{(c)}\|_{L_t^{\frac{12}{1-2\epsilon}}W^{2\epsilon,\frac{3}{1+\epsilon}}}\Big)$$
$$\times\Big(\|\phi_\ell^{(c)}\|_{L_t^\infty H^s}+\|\phi_k^{(c)}\|_{L_t^\infty H^s}\Big).$$

Collecting all, we conclude that

$$\|\Gamma(\Phi^{(c)})\|_{X_I}\le A\|\Phi_0\|_{L^2(\mathbb{R}^3;\mathbb{C}^N)}$$
$$+4A|\lambda|CT^{\frac{5+2\epsilon}{6}}\sum_{k,\ell=1}^N\Big\{\Big(\|\phi_\ell^{(c)}\|_{L_t^{\frac{12}{1-2\epsilon}}W^{2\epsilon,\frac{3}{1+\epsilon}}}+\|\phi_k^{(c)}\|_{L_t^{\frac{12}{1-2\epsilon}}W^{2\epsilon,\frac{3}{1+\epsilon}}}\Big)$$
$$\times\|\phi_\ell^{(c)}\|_{L_t^{\frac{12}{1-2\epsilon}}W^{2\epsilon,\frac{3}{1+\epsilon}}}\Big(\|\phi_\ell^{(c)}\|_{L_t^\infty H^s}+\|\phi_k^{(c)}\|_{L_t^\infty H^s}\Big)\Big\}$$
$$\le A\|\Phi_0\|_{L^2(\mathbb{R}^3;\mathbb{C}^N)}+4A|\lambda|CT^{\frac{5+2\epsilon}{6}}\|\Phi^{(c)}\|^3_{X_I},$$

for some C independent of $c\ge 1$. Analogously, we obtain such an a-priori bound for difference of two solutions $\Phi_1^{(c)},\Phi_2^{(c)}\in X_{I,A}$ that

$$\begin{aligned}&\|\Gamma(\Phi_1^{(c)})-\Gamma(\Phi_2^{(c)})\|_{X_I}\\&\quad\le 4A|\lambda|CT^{\frac{5+2\epsilon}{6}}(\|\Phi_1^{(c)}\|_{X_I}+\|\Phi_2^{(c)}\|_{X_I})^2\|\Phi_1^{(c)}-\Phi_2^{(c)}\|_{X_I}.\end{aligned}\tag{3.3}$$

By taking an appropriate small T depending on $\|\Phi_0\|_{H^s(\mathbb{R}^3;\mathbb{C}^N)}$, the map Γ is a contraction map in $X_{I,A}$, thus we can find the solution $\Phi^{(c)}$ satisfying

$$\|\Phi^{(c)}\|_{X_I} \le 4A\|\Phi_0\|_{H^s(\mathbb{R}^3;\mathbb{C}^N)}. \tag{3.4}$$

3.2. Proof of Theorem 1.6: Higher-order approximation when J is even

3.2.1. *Results on pseudo-relativistic equations.* The well-posedness results for (hHF) on the Sobolev space $H^s(\mathbb{R}^3)$ have been established in [7, 11, 14, 12]. In particular, the proof of the local well-posedness in $H^s(\mathbb{R}^3)$ for $s \ge \frac{1}{2}$ follows from the standard contraction mapping principle only with the help of Sobolev embedding, which implies that the solutions are bounded in uniformly in c. The statement of the result (without proof) is as follows:

Proposition 3.1. *Let $c \ge 1$. Suppose that $\Psi_0 = \{\psi_{k,0}\}_{k=1}^N \in H^s(\mathbb{R}^3;\mathbb{C}^N)$ for $s \ge \frac{1}{2}$. Then, there exists a time $T > 0$, independent of $c > 1$, such that there exists a local solutions $\Psi^{(c)} = \{\psi_k^{(c)}(t)\}_{k=1}^N \in C([-T,T];H^s(\mathbb{R}^3;\mathbb{C}^N))$ to* (hHF) *such that*

$$\sup_{t\in[-T,T]} \|\Psi^{(c)}(t)\|_{H^s(\mathbb{R}^3;\mathbb{C}^N)} \lesssim \|\Psi_0\|_{H^s(\mathbb{R}^3;\mathbb{C}^N)}, \tag{3.5}$$

where the implicit constant is independent of $c > 1$.

Remark 3.2. In Proposition 3.1, the assumption on regularity $s \ge \frac{1}{2}$ is necessary to obtain solutions uniformly bounded in $c \ge 1$. In fact, by allowing the dependence on c, the regularity assumption for local well-posedness of (hHF) can be improved by just using the dispersive property, namely, Strichartz estimates for (pLS). For example, in [7], the authors proved the local well-posedness of (hHF) for $s > \frac{1}{3}$. However, we cannot lower the regularity assumption to below $\frac{1}{2}$ if we persist in obtaining solutions which have uniform bounds in $c \ge 1$, even though we employ the following uniform-in-c Strichartz estimates because of an additional loss in $c \ge 1$ besides the regularity loss.

Lemma 3.3 (Lemma 2.1 in [19]). *For $2 \le q \le \infty$, $2 \le r < \infty$ such that $\frac{1}{q} + \frac{1}{r} = \frac{1}{2}$, we have*

$$\left\| e^{-it\mathcal{H}^{(c)}} \psi_0 \right\|_{L_t^q(\mathbb{R};L_x^r(\mathbb{R}^3))} \le A c^{\frac{1}{q}} \|\psi_0\|_{H^{\frac{2}{q}}(\mathbb{R}^3)}, \tag{3.6}$$

where A is independent of $c > 1$.

Indeed, to obtain uniformly bounded solutions via the contraction mapping argument, one also has to bound $c^{\frac{1}{q}}$ in the left-hand side of (3.6), which requires the regularity, $s > \frac{1}{2}$, of solutions, not just $s > \frac{1}{3}$.

3.2.2. *Approximation.* We can easily generalize the result in [13, Lemma 5.4], with a slight modification, to a function in any Sobolev spaces as follows:

LEMMA 3.4. *Let $c \geq 1$, $t \in \mathbb{R}$ and $s \geq 0$. For any $f \in H^s(\mathbb{R}^3)$, we have*

$$\|(e^{-it\mathcal{H}^{(c)}} - e^{-it\mathcal{H}_J^{(c)}})f\|_{L^2(\mathbb{R}^3)} \lesssim c^{-\frac{Js}{J+1}}\langle t\rangle\|f\|_{H^s(\mathbb{R}^3)},$$

where the implicit constant is independent of $c \geq 1$.

Now we are ready to prove the Theorem 1.6.

Proof of Theorem 1.6. We write the difference of (hHF) and (pHF) with $\Psi_0 = \Phi_0$ as

$$\begin{aligned}
\psi_k^{(c)}(t) - \phi_k^{(c)}(t) &= (e^{-it\mathcal{H}^{(c)}} - e^{-it\mathcal{H}_J^{(c)}})\psi_{k,0} \\
&\quad - i\int_0^t \left(e^{-i(t-s)\mathcal{H}^{(c)}} - e^{-i(t-s)\mathcal{H}_J^{(c)}}\right)\left(H(\psi_k^{(c)}) - F_k(\psi_k^{(c)})\right)(s)\; ds(s)ds \\
&\quad + i\kappa\int_0^t e^{-i(t-s)\mathcal{H}_J^{(c)}}\mathcal{N}_1(\psi_k^{(c)}, \phi_k^{(c)})(s)\; ds \\
&\quad + i\kappa\int_0^t e^{-i(t-s)\mathcal{H}_J^{(c)}}\mathcal{N}_2(\psi_k^{(c)}, \phi_k^{(c)})(s)\; ds \\
&=: \mathcal{I} + \mathcal{II} + \mathcal{III} + \mathcal{IV},
\end{aligned}$$

for $k = 1, 2, \cdots, N$, where

$$\begin{aligned}
\mathcal{N}_1(\psi_k^{(c)}, \phi_k^{(c)}) =& \sum_{\ell=1}^N \left(|x|^{-1} * (|\psi_\ell^{(c)}|^2 - |\phi_\ell^{(c)}|^2)\right)\phi_k^{(c)} \\
&- \sum_{\ell=1}^N \left(|x|^{-1} * (\overline{\psi_\ell^{(c)}}\psi_k^{(c)} - \overline{\phi_\ell^{(c)}}\phi_k^{(c)})\right)\phi_\ell^{(c)}, \\
\mathcal{N}_2(\psi_k^{(c)}, \phi_k^{(c)}) =& \sum_{\ell=1}^N \left(|x|^{-1} * (|\psi_\ell^{(c)}|^2)\right)(\psi_k^{(c)} - \phi_k^{(c)}) \\
&- \sum_{\ell=1}^N \left(|x|^{-1} * (\overline{\psi_\ell^{(c)}}\psi_k^{(c)})\right)(\psi_\ell^{(c)} - \phi_\ell^{(c)}).
\end{aligned}$$

By Lemma 3.4, we immediately obtain

$$\|\mathcal{I}\|_{L^2(\mathbb{R}^3)} \lesssim c^{-\frac{Js}{J+1}}\langle t\rangle \|\psi_{k,0}\|_{H^s(\mathbb{R}^3)}. \tag{3.7}$$

Moreover, by using Lemma 3.4, (2.5) and Proposition 3.1, we have

$$\begin{aligned}\|\mathcal{II}\|_{L^2(\mathbb{R}^3)} &\lesssim c^{-\frac{Js}{J+1}} \int_0^t \langle t-s\rangle \left\| H(\psi_k^{(c)})(s) - F_k(\psi_k^{(c)})(s) \right\|_{H^s(\mathbb{R}^3)} ds \\ &\lesssim c^{-\frac{Js}{J+1}}\langle t\rangle^2 \sup_{s\in[0,t]} \|\psi_k^{(c)}(s)\|_{H^s(\mathbb{R}^3)}^3 \lesssim c^{-\frac{Js}{J+1}}\langle t\rangle^2 \|\psi_{k,0}\|_{H^s(\mathbb{R}^3)}^3.\end{aligned} \tag{3.8}$$

Furthermore, by (2.6) and the Sobolev embedding $H^s(\mathbb{R}^3) \hookrightarrow L^3(\mathbb{R}^3)$ for $s \geq \frac{1}{2}$, we have

$$\begin{aligned}\|\mathcal{III}\|_{L^2(\mathbb{R}^3)} &\lesssim \int_0^t \left\| \mathcal{N}_1(\psi_k^{(c)}, \phi_k^{(c)})(s) \right\|_{L^2(\mathbb{R}^3)} ds \\ &\lesssim \int_0^t \left(\|\Psi^{(c)}(s)\|_{H^s(\mathbb{R}^3)} + \|\Phi^{(c)}(s)\|_{H^s(\mathbb{R}^3)} \right) \|(\Psi^{(c)} - \Phi^{(c)})(s)\|_{L^2(\mathbb{R}^3)} \\ &\qquad \times \left(\|\psi_k^{(c)}(s)\|_{H^s(\mathbb{R}^3)} + \|\phi_k^{(c)}(s)\|_{H^s(\mathbb{R}^3)} \right) ds \\ &\quad + \int_0^t \left(\|\Psi^{(c)}(s)\|_{H^s(\mathbb{R}^3)}^2 + \|\Phi^{(c)}(s)\|_{H^s(\mathbb{R}^3)}^2 \right) \|(\psi_k^{(c)} - \phi_k^{(c)})(s)\|_{L^2(\mathbb{R}^3)} ds\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}\|\mathcal{IV}\|_{L^2(\mathbb{R}^3)} &\lesssim \int_0^t \left\| \mathcal{N}_2(\psi_k^{(c)}, \phi_k^{(c)})(s) \right\|_{L^2(\mathbb{R}^3)} ds \\ &\lesssim \int_0^t \|\Psi^{(c)}(s)\|_{H^s(\mathbb{R}^3)}^2 \|(\psi_k^{(c)} - \phi_k^{(c)})(s)\|_{L^2(\mathbb{R}^3)} ds \\ &\quad + \int_0^t \|\Psi^{(c)}(s)\|_{H^s(\mathbb{R}^3)} \|(\Psi^{(c)} - \Phi^{(c)})(s)\|_{L^2(\mathbb{R}^3)} \|\psi_k^{(c)}\|_{H^s(\mathbb{R}^3)} ds\end{aligned} \tag{3.10}$$

respectively. Thus, by collecting (3.7)–(3.10), and by applying Proposition 3.1, we conclude that

$$\begin{aligned}&\|\Psi^{(c)}(t) - \Phi^{(c)}(t)\|_{L^2(\mathbb{R}^3)} \\ &\lesssim c^{-\frac{Js}{J+1}}\langle t\rangle^2 \|\Psi_0\|_{H^s(\mathbb{R}^3)} \left(1 + \|\Psi_0\|_{H^s(\mathbb{R}^3)}^2\right) \\ &+ \int_0^t \left(\|\Psi_0(s)\|_{H^s(\mathbb{R}^3)}^2 + \|\Phi^{(c)}(s)\|_{H^s(\mathbb{R}^3)}^2 \right) \|(\Psi^{(c)} - \Phi^{(c)})(s)\|_{L^2(\mathbb{R}^3)} ds.\end{aligned}$$

By Gronwall's inequality, in addition to (1.5), we have

$$\|\Psi^{(c)}(t) - \Phi^{(c)}(t)\|_{L^2(\mathbb{R}^3)} \lesssim c^{-\frac{Js}{J+1}} A e^{Bt},$$

here the constants A, B depend on $\|\Psi_0\|_{H^s(\mathbb{R}^3)}$, but not c. □

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Changhun Yang
Department of Mathematics
Chungbuk National University, Cheongju 28644, Korea
E-mail: chyang@chungbuk.ac.kr